

## Gravitational Bremsstrahlung from Massless-particle Collisions

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The angular and frequency characteristics of the gravitational radiation emitted in collisions of massless particles is studied perturbatively in the context of classical General Relativity for small values of the ratio  $\alpha \equiv 2r_S/b$  of the Schwarzschild radius over the impact parameter. The particles are described with their trajectories, while the contribution of the leading nonlinear terms of the gravitational action is also taken into account. The old quantum results are reproduced in the zero frequency limit  $\omega \ll 1/b$ . The radiation efficiency  $\epsilon \equiv E_{\text{rad}}/2E$  outside a narrow cone of angle  $\alpha$  in the forward and backward directions with respect to the initial particle trajectories is given by  $\epsilon \sim \alpha^2$  and is dominated by radiation with characteristic frequency  $\omega \sim \mathcal{O}(1/r_S)$ . The comparison with previous works and the known literature is presented.

### 1. Introduction

The problem of gravitational radiation in particle collisions has a long history and has been studied in a variety of approaches and approximations. The problem of gravitational radiation in *massless*-particle collisions is worth studying in its own right and has attracted the interest of many authors in the past as well as very recently. Apart from its obvious relevance in the context of TeV-scale gravity models with large extra dimensions<sup>2</sup>, it is very important in relation to the structure of string theory and the issue of black-hole formation in ultra-planckian collisions<sup>3</sup>. Nevertheless, to the best of our knowledge, complete understanding of all facets of the problem is still lacking. The emission of radiation in the form of *soft* gravitons was computed in<sup>4</sup> in the context of quantum field theory, but in that computation the contribution of the non-linear graviton self-couplings i.e. the stress part of the energy-momentum tensor, was argued to be negligible. The result of the quantum computation for low-frequency graviton emission was reproduced by a purely classical computation in<sup>5</sup>. More recently, a new approach was put forward for the computation of the characteristics of the emitted radiation,<sup>6</sup> based on the Fraunhofer approximation of radiation theory.

The purpose here is to extend the method used in<sup>1</sup> to the study of gravitational radiation in collisions of massless particles with center-of-mass energy  $2E$  and impact parameter  $b$ . The formal limit  $m \rightarrow 0$  (or equivalently  $\gamma_{\text{cm}} \rightarrow \infty$  for the Lorentz factor) of the massive case leads to nonsensical answers for the radiation efficiency, i.e. the ratio  $\epsilon \equiv E_{\text{rad}}/2E \sim (r_S/b)^3 \gamma_{\text{cm}}$  of the radiated to the available energy, the characteristic radiation frequency  $\omega \sim \gamma_{\text{cm}}^2/b$ , or the characteristic emission angle  $\vartheta \sim 1/\gamma_{\text{cm}}$ . We study classically the gravitational radiation in the collision of massless particles using the same perturbative approach as in<sup>1</sup>.

This presentation is mainly based on results obtained in<sup>7</sup>.

## 2. Scattering

The action describing the two massless particles and their gravitational interaction reads

$$S = -\frac{1}{2} \sum \int e(\sigma) g_{\mu\nu}(z(\sigma)) \dot{z}^\mu(\sigma) \dot{z}^\nu(\sigma) d\sigma - \frac{1}{\kappa^2} \int R \sqrt{-g} d^4x, \quad (1)$$

where  $e(\sigma)$  is the einbein of the trajectory  $z^\mu(\sigma)$  in terms of the corresponding affine parameter  $\sigma$ ,  $\kappa^2 = 16\pi G$  and the summation is over the two particles. We will be using unprimed and primed symbols to denote quantities related to the two particles. For identical colliding particles in the center-of-mass frame  $e = \sqrt{s}/2 = E$ , with  $E$  the energy of each colliding particle.

Thus, the particles move on null geodesics, while variation of  $z^\mu$  leads to the particle equation of motion:

$$\frac{d}{d\sigma} (g_{\mu\nu} \dot{z}^\nu) = \frac{1}{2} g_{\lambda\nu, \mu} \dot{z}^\lambda \dot{z}^\nu \quad (2)$$

and similarly for  $z'^\mu$ . The particle energy-momentum is defined by  $T^{\mu\nu} \equiv (-2/\sqrt{-g}) \delta S / \delta g_{\mu\nu}$ . At zeroth order in the gravitational interaction, the space-time is flat and the particles move on straight lines with constant velocities, i.e.  ${}^0g_{\mu\nu} = \eta_{\mu\nu}$ ;  ${}^0\dot{z}^\mu \equiv u^\mu = (1, 0, 0, 1)$ ,  ${}^0\dot{z}'^\mu \equiv u'^\mu = (1, 0, 0, -1)$ . Given that  ${}^0T_{\mu\nu}$  is traceless, the perturbation  $h_{\mu\nu}$  satisfies for each particle separately the equation  $\partial^2 h_{\mu\nu} = -\kappa^2 T_{\mu\nu}$ , whose solution in coordinate representation is the Aichelburg – Sexl metric

$$h_{\mu\nu}(x) = -\kappa e u_\mu u_\nu \delta(t - z) \Phi(|\mathbf{r} - \mathbf{b}/2|)$$

where  $\Phi$  is the 2-dimensional Fourier transform of  $1/q^2$ :  $\Phi(r) = -(1/2\pi) \ln(r/r_0)$  with  $r_0$  an arbitrary constant length parameter.

Write for the metric  $g_{\mu\nu} = \eta_{\mu\nu} + \kappa(h_{\mu\nu} + h'_{\mu\nu})$  and substitute in (2) to obtain for the first correction of the trajectory of the unprimed particle the equation

$${}^1\ddot{z}_\mu(\sigma) = -\kappa \left( h'_{\mu\nu, \lambda} - \frac{1}{2} h'_{\lambda\nu, \mu} \right) {}^0\dot{z}^\lambda {}^0\dot{z}^\nu. \quad (3)$$

The interaction with the self-field of the particle has been omitted and  $h'_{\mu\nu}$  due to the primed particle is evaluated at the location of the unprimed particle on its unperturbed trajectory.

Integrating (3) one obtains the trajectory in components:

$${}^1z^0(\sigma) = \frac{1}{2} e \kappa^2 \Phi(b) \theta(\sigma) = -{}^1z^z(\sigma), \quad {}^1z^x(\sigma) = e \kappa^2 \Phi'(b) \sigma \theta(\sigma). \quad (4)$$

which vanish for all  $\sigma < 0$ . Indeed, the massless particle trajectories should remain undisturbed before the collision. Thus, we reproduce the leading order expressions of the two well-known facts<sup>8</sup> about the geodesics in an Aichelburg-Sexl metric, namely: (i) the *time delay* at the moment of shock equal to  $\Delta t = e \kappa^2 \Phi(b) = 8GE \ln(b/r_0)$ ; (ii) the *refraction* caused by the gravitational interaction by an angle  $\alpha = e \kappa^2 |\Phi'(b)| = 8GE/b$  in the direction of the center of gravity.

<sup>\*</sup>The upper left index on a symbol labels its order in our perturbation scheme.

### 3. Radiation amplitude

The gravitational wave source has two parts. One is the particle energy-momentum contribution, localized on the accelerated particle trajectories.

The direct particle contribution to the source of radiation is called “local”, because it is localized on the particle trajectories. The first order term in the expansion of  $T_{\mu\nu}$  is

$${}^1T_{\mu\nu}(x) = e \int d\sigma \left[ 2 {}^1\dot{z}_{(\mu} u_{\nu)} + 2 \mathcal{K} u^\lambda h'_{\lambda(\mu} u_{\nu)} - u_\mu u_\nu ({}^1z \cdot \partial) \right] \delta^4(x - {}^0z(\sigma)), \quad (5)$$

where  $z^\mu$  is evaluated at  $\sigma$  and  $h'_{\mu\nu}$  is evaluated at  ${}^0z^\mu(\sigma)$ .

In Fourier space one obtains effectively

$${}^1T_{\mu\nu} = -2e^2 \mathcal{K}^2 e^{i(kb)/2} \left[ -\Phi(b) u_\mu u_\nu + \frac{(ku') \Phi(b)}{2(ku)} u_\mu u_\nu + i \frac{\Phi'(b) \sigma_{\mu\nu}^{(u)}}{b(ku)^2} \right]. \quad (6)$$

The contribution to the source at second-order coming from the expansion of the Einstein tensor, reads<sup>1</sup>

$$S_{\mu\nu} = h_\mu^{\lambda,\rho} (h_{\nu\rho,\lambda} - h_{\nu\lambda,\rho}) + h^{\lambda\rho} (h_{\mu\lambda,\nu\rho} + h_{\nu\lambda,\mu\rho} - h_{\lambda\rho,\mu\nu} - h_{\mu\nu,\lambda\rho}) - \\ - \frac{1}{2} h^{\lambda\rho}_{,\mu} h_{\lambda\rho,\nu} - \frac{1}{2} h_{\mu\nu} \partial^2 h + \frac{1}{2} \eta_{\mu\nu} \left( 2 h^{\lambda\rho} \partial^2 h_{\lambda\rho} - h_{\lambda\rho,\sigma} h^{\lambda\sigma,\rho} + \frac{3}{2} h_{\lambda\rho,\sigma} h^{\lambda\rho,\sigma} \right).$$

Upon substitution of  $h_{\mu\nu}$  and  ${}^1z^\mu(\sigma)$  of the previous section in the above expression we obtain for the Fourier transform of  $S_{\mu\nu}$

$$S_{\mu\nu}(k) = \mathcal{K}^2 e^2 e^{i(kb)/2} \left[ (ku')^2 u_\mu u_\nu J + (ku)^2 u'_\mu u'_\nu J + 4 J_{\mu\nu} + 4(ku') u_{(\mu} J_{\nu)} - \right. \\ \left. - 4(ku) u'_{(\mu} J_{\nu)} + 2 u_{(\mu} u'_{\nu)} \left( 2(kJ) - (ku)(ku') J - 2 \text{Sp} J \right) \right]$$

in terms of the integrals ( $l = 0, 1, 2$ )

$$J_{\mu_1 \dots \mu_l}(k) \equiv \frac{1}{(2\pi)^2} \int \frac{\delta(qu') \delta(ku - qu) e^{-i(qb)}}{q^2(k - q)^2} q_{\mu_1} \dots q_{\mu_l} d^4q.$$

**Total radiation amplitudes.** Defining in the center-of-mass frame the radiation wave-vector by  $k^\mu = \omega(1, \mathbf{n}) = \omega(1, \sin \vartheta \cos \varphi, \sin \vartheta \sin \varphi, \cos \vartheta)$  and contracting with the two polarizations (defined as usual), we obtain the final (finite for  $\vartheta > 0$ ) expressions for the source of the gravitational radiation separately for the two polarizations (here  $\hat{K}_\nu(\zeta) \equiv \zeta^\nu K_\nu(\zeta)$ ,  $\zeta = \omega b \sin \vartheta \sqrt{x(1-x)}$ ):

$$\tau_+(k) = \frac{16Ge^2}{\sqrt{2}} \int_0^1 dx e^{-i(kb)x} \left[ -K_0(\zeta) + \sin^2 \varphi \hat{K}_1(\zeta) \right], \quad (7) \\ \tau_\times(k) = -\frac{16Ge^2}{\sqrt{2}} \sin \varphi \int_0^1 dx e^{-i(kb)x} \left[ 2i \frac{\hat{K}_2(\zeta) - \hat{K}_1(\zeta)}{\omega b \sin \vartheta} + (2x - 1) \cos \varphi \hat{K}_1(\zeta) \right].$$

#### 4. Characteristics of the emitted radiation

The emitted radiation frequency spectrum and of the total emitted energy are obtained from (sum over the two polarizations):

$$\frac{dE_{\text{rad}}}{d\omega d\Omega} = \frac{G}{2\pi^2} \omega^2 \sum_{\mathcal{P}} |\tau_{\mathcal{P}}|^2. \quad (8)$$

It will be convenient in the sequel to treat separately the six angular and frequency regimes shown in Fig. 1.

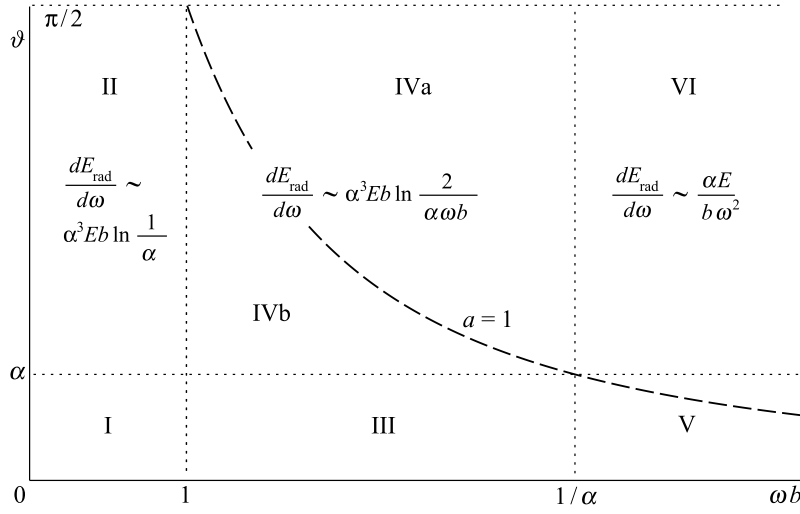


Fig. 1. The characteristic angular and frequency regimes ( $a \equiv \omega b \sin \vartheta$ ).

**Zero-frequency limit.** In the low-frequency regime ( $\omega \rightarrow 0$ ) the amplitude  $\tau_{\times}$  dominates and has the form

$$\tau_{\times} \simeq -\frac{16\sqrt{2}iGE^2 \sin \varphi}{\omega b \sin \vartheta}, \quad (9)$$

while  $\tau_{+}$  is finite (as  $\vartheta \rightarrow 0$ ) and gives subleading contribution to (8). The angular integration leads to  $dE_{\text{rad}}/d\omega = (2^8 G^3 E^4 / \pi b^2) \int d\vartheta / \sin \vartheta$ , which diverges and implies that our formulae are not valid for  $\vartheta$  close to zero.

We cannot trust our formulae in regime I and should repair them. A quick way to do it, is to impose a small-angle cut-off  $\vartheta_{\min} = \alpha$  on the  $\vartheta$ -integration, so as to obtain for  $dE_{\text{rad}}/d\omega|_{\omega=0}$  the value computed quantum mechanically in <sup>4,5</sup>:

$$\left( \frac{dE^W}{d\omega} \right)_{\omega=0} = \frac{4}{\pi} G|t| \ln \frac{s}{|t|} = \frac{G\alpha^2 s}{\pi} \ln \frac{4}{\alpha^2} = \frac{Eb}{\pi} \alpha^3 \ln \frac{2}{\alpha}, \quad s = 4E^2, \quad (10)$$

which agrees with our expression for  $\vartheta > \alpha$ . Thus, in regime II we use our formula, which is identical to Weinberg's. Now for any frequency we have to compare our matter part, which makes the  $\vartheta$ -integration divergent, with the Weinberg's:

$$\tilde{T}_{\text{eff}}^{\mu\nu}(k) = i \sum_{n=1}^2 e^{\pm i(kb)/2} \left( \frac{\tilde{P}_n^\mu \tilde{P}_n^\nu}{\tilde{E}_n} \frac{1}{\omega - \mathbf{k}\tilde{\mathbf{v}}_n} - \frac{P_n^\mu P_n^\nu}{E_n} \frac{1}{\omega - \mathbf{k}\mathbf{v}_n} \right). \quad (11)$$

To leading order in our approximation the scattering process, we are dealing with, is elastic with  $\tilde{E}_n = E_n = E$ . Write for the incoming particles  $P_n^\mu = Eu_n^\mu = E(1, 0, 0, \pm 1)$  and for the outgoing ones  $\tilde{P}_n^\mu = E\tilde{u}_n^\mu = E(u_n^\mu + \hat{z}_n^\mu) = E(u_n^\mu \mp \alpha \hat{b}^\mu) \equiv P_n^\mu + {}^1P_n^\mu$ , substitute into  $\tilde{T}_{\text{eff}}^{\mu\nu}$ , and expand in powers of  $\alpha$  to have  $|k \cdot {}^1P_n^\mu| \ll |k \cdot P_n^\mu|$ , to obtain the tensor which is *identical* to the local part  $T_{\mu\nu}$  we use, at *any* frequency.

Therefore, it is natural to impose the angular cutoff  $\vartheta > \alpha$  for all frequencies of the emitted graviton.

**Regime VI.** For  $a \gg 1$  the two radiation amplitudes read:

$$\tau_+(k) \approx -\frac{64Ge^2}{\sqrt{2}a^2} \cos 2\varphi \cos \frac{a \cos \varphi}{2}, \quad \tau_\times(k) \approx -\frac{64iGe^2}{\sqrt{2}a^2} \sin 2\varphi \sin \frac{a \cos \varphi}{2},$$

from which one can obtain an estimate for the frequency distribution of the emitted radiation in regime VI by integrating over  $\varphi$  and over  $\vartheta \in (\alpha, \pi - \alpha)$ , namely

$$\frac{dE_{\text{rad}}^{\text{VI}}}{d\omega} \sim \frac{\alpha E}{b} \frac{1}{\omega^2}, \quad \omega > 1/\alpha b, \quad (12)$$

as well as an estimate for the emitted energy and the corresponding efficiency in regime VI, by integrating also over  $\omega \in (1/\alpha b, \infty)$ ,

$$E_{\text{rad}}^{\text{VI}} \sim \alpha^2 E \quad \text{and} \quad \epsilon_{\text{VI}} \sim \alpha^2. \quad (13)$$

**Regime IV.** Inside the regime IVa the amplitude is damped as in regime VI. However, near the left border of regime IVb (with  $1/b \lesssim \omega \ll 1/\alpha b$ ) one may expand the amplitudes in powers of  $a$  and obtain (9). Upon integration over regime IVb, i.e. for  $\alpha \lesssim \vartheta \lesssim \vartheta_{\text{max}} = \arcsin(1/\omega b)$ , one obtains

$$\left( \frac{dE_{\text{rad}}^{\text{IV}}}{d\omega} \right)_{1/b \lesssim \omega \ll 1/\alpha b} \simeq \frac{\alpha^3 E b}{\pi} \ln \frac{\text{tg}(\vartheta_{\text{max}}/2)}{\text{tg}(\alpha/2)} \simeq \frac{\alpha^3 E b}{\pi} \ln \frac{2\alpha^{-1}}{\omega b + \sqrt{\omega^2 b^2 - 1}}.$$

Thus, for  $1/b \lesssim \omega \ll 1/\alpha b$  one may approximate  $dE_{\text{rad}}/d\omega$  by

$$\left( \frac{dE_{\text{rad}}^{\text{IV}}}{d\omega} \right)_{1/b \lesssim \omega \lesssim 1/\alpha b} \sim \alpha^3 E b \ln \frac{2}{\alpha \omega b}. \quad (14)$$

It should be pointed out here that the integral of  $dE_{\text{rad}}/d\omega$  over  $\omega$  receives most of its contribution from frequencies in the neighborhood of  $1/\alpha b$  in both regimes IV and VI. Thus, one can say that the characteristic frequency of the emitted radiation is around  $\mathcal{O}(1/r_S)$ .

**Total emitted radiation.** Upon integration (9) leads to the angular distribution

$$\frac{dE_{\text{rad}}}{d\vartheta} = \frac{\eta\alpha^3 E}{8\pi^2} \frac{1}{\sin^2\vartheta}. \quad (15)$$

Integration over  $\vartheta \in (\alpha, \pi - \alpha)$  gives

$$E_{\text{rad}} = \frac{\eta\alpha^2 E}{4\pi^2}, \quad \epsilon = \frac{E_{\text{rad}}}{2E} \simeq 1.14\alpha^2. \quad (16)$$

## 5. Conclusions – Discussion

Using the same approach as in<sup>1</sup>, based on standard GR, with the leading non-linear gravity effects taken into account, we studied collisions of massless particles and computed the gravitational energy of arbitrary frequency, which is emitted outside the cone of angle  $\alpha = 2r_S/b \ll 1$  in the forward and backward directions. The value  $\epsilon \simeq 1.14\alpha^2$  was obtained for the radiation efficiency, with characteristic frequency  $\omega \sim 1/r_S$ . In fact, this value represents a lower bound of the efficiency, since it does not include the energy emitted inside that cone. The frequency distribution of radiation in the characteristic angle-frequency regimes is shown in Fig. 1. Unfortunately, we cannot yet confirm the presence of any other characteristic frequency<sup>6</sup>, such as e.g.  $1/\alpha^3 b$ , or characteristic emission angle smaller than  $\alpha$ . We hope to return to these issues with a better understanding of regime V in the near future.

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